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Quantum dynamics of a two-level system coupled to a bosonic degree of freedom in terms of Wigner phase space distributions

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Abstract. The quantum dynamics of the Hamiltonian of a two-level system coupled to a boson mode is formulated in terms of the Wigner matrix. The trace of the Wigner matrix gives complete information for the bosonic degree of freedom. The time evolution and the long time average of the trace of the Wigner matrix is calculated numerically for small perturbations and in resonance. Some aspects of 'quantum chaos' of the two-level system are discussed.

1. Introduction

In a recent paper (Klenner *et al* 1986, hereafter referred to as I) we have discussed the quantum and semiclassical dynamics of the Hamiltonian

$$H = b^+b + \sqrt{2}\kappa\sigma_x(b + b^+) + \bar{\delta}\sigma_z. \tag{1.1}$$

We have shown that the semiclassical dynamics is not at all appropriate to approximate the quantum dynamics for the parameter ranges which were considered. These parameter ranges are not relevant for quantum optical applications but are well suited to discuss the two-site model of small polaron motion.

In I, we have worked out a suitable method to treat the eigenvalue problem of the Hamiltonian (1.1). We will use the same method in this paper to discuss the quantum dynamics of the Hamiltonian (1.1) in terms of Wigner phase space distributions. In particular, we will use the same complete set of orthonormal functions in Bargmann space as in I to expand the eigenfunctions of the Hamiltonian (1.1). We are interested in the quantum dynamics of (1.1) for two reasons. Firstly, although (1.1) is a basic model in various fields, dynamical calculations are still rare (Eberly *et al* 1980, Graham and Höhnerbach 1984a). The dynamics of a simple non-adiabatic model Hamiltonian perhaps shows some qualitative features that are also important for more complicated and more realistic non-adiabatic Hamiltonians. Secondly, the semiclassical dynamics of (1.1) is chaotic (Belobrov *et al* 1976, Milonni *et al* 1983, Feinberg and Ranninger 1984, Ackerhalt *et al* 1985).

For this reason, the two-level system was considered to be a simple example of 'quantum chaos' (Graham and Höhnerbach 1984b). In this paper we will discuss some of the qualitative features of the quantum motion of the two-level system. In § 3 of I, we presented the dynamics of the fermion in detail. We have calculated the expectation values $\langle\psi(t)|\sigma_x|\psi(t)\rangle$, $\langle\psi(t)|\sigma_y|\psi(t)\rangle$, $\langle\psi(t)|\sigma_z|\psi(t)\rangle$ and the entropy of the fermion $S_F(t)$ starting with a product state (equation (3.2) of I). This gives a clear picture of the dynamics of the fermion degree of freedom.

On the other hand, we have calculated the expectation values $\langle\psi(t)|\hat{q}|\psi(t)\rangle$ and $\langle\psi(t)|\hat{p}|\psi(t)\rangle$ for the bosonic degree of freedom. But this is not sufficient to get a clear picture of the bosonic dynamics, since very different bosonic wavefunctions can give the same bosonic expectation values. The purpose of this paper is to present the bosonic dynamics in more detail. We will use the Wigner-Weyl representation of the quantum mechanics to describe the bosonic behaviour since it gives information about configuration *and* momentum of the bosonic degree of freedom. In the classical limit ($\hbar \to 0$) the dynamics of the Wigner phase space distribution reduces to the phase space dynamics of the corresponding classical Hamiltonian. Since the phase space is usually used to discuss classical chaos, the Wigner phase space distribution should be appropriate to discuss 'quantum chaos'. Such an analysis was carried out for the Henon-Heiles Hamiltonian (Hutchinson and Wyatt 1980) and for a periodically kicked particle (Korsch and Berry 1981). Quite recently, a quantum rotator kicked periodically by δ pulses was analysed in the Wigner-Weyl representation for the case of angle-action variables (Berman and Kolovsky 1985). Usually the systems under consideration are systems with two degrees of freedom or time-dependent systems with one degree of freedom. These are the simplest systems where classically chaotic motion can arise. All the systems mentioned above are of this type. The Hamiltonian (1.1) is, in some respects, not of this type. Clearly we have two degrees of freedom, but the Hilbert space of the fermion is only two dimensional. This drastically simplifies the calculations and leads to the concept of a Wigner matrix.

The paper is organised as follows. In § 2 we present some basic facts about the Wigner-Weyl representation of one-dimensional quantum systems. We will point out some properties of the Wigner phase space distribution which will be needed for the interpretation of the results presented in § 4. In § 3 we introduce the Wigner matrix and discuss the relevance of its trace for the bosonic degree of freedom. We will see that the diagonal elements of the Wigner matrix give restricted information about the fermion whereas the trace of the Wigner matrix gives complete information about the bosonic degree of freedom. Further we write down the Schrödinger equation in the Wigner matrix formulation to point out some peculiarities of the two-level system. In § 4, we present the numerical results for the Wigner matrix. We discuss the dynamics of the trace of the Wigner matrix for small perturbation ($\bar{\delta}=0.1$) and resonance ($\bar{\delta}=0.5$). It seems that the dynamics is similar in both cases when evolving on different timescales up to times for which $\bar{\delta}t<10$. Additionally we have calculated the long time average of the trace of the Wigner matrix in both cases. Finally, we discuss some aspects of 'quantum chaos' of the two-level system in § 5.

2. Wigner-Weyl representation for quantum systems with one degree of freedom

The Wigner-Weyl formalism is described clearly in the book of de Groot and Suttorp (1972) and we will use their notation here. The Weyl transform $a(q,p)$ of an operator $\hat{A}(\hat{q},\hat{p})$ is a real function in phase space and is defined by

$$a(q,p)=\int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}pv}\langle q-\tfrac{1}{2}v|\hat{A}(\hat{q},\hat{p})|q+\tfrac{1}{2}v\rangle. \tag{2.1}$$

In our units we have $\hbar=1$. A quantum system is usually described by its state vector $|\psi(t)\rangle$ or by the density matrix $\hat{\rho}=|\psi(t)\rangle\langle\psi(t)|$. But one can equally well describe the quantum system by the Weyl transform of the density matrix $\hat{\rho}$. The Wigner phase space distribution defined by

$$W(q,p,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}v\,\mathrm{e}^{ipv}\langle q-\tfrac{1}{2}v|\psi(t)\rangle\langle\psi(t)|q+\tfrac{1}{2}v\rangle \tag{2.2}$$

is (modulo 2π) the Weyl transform of the density operator. It is important to note that every expectation value of an arbitrary operator $\hat{A}(\hat{q},\hat{p})$ can be expressed in terms of its Weyl transform $a(q,p)$ and the Wigner phase space distribution $W(q,p,t)$ by

$$\langle\psi(t)|\hat{A}(\hat{q},\hat{p})|\psi(t)\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{d}p\,a(q,p)\,W(q,p,t). \tag{2.3}$$

For this reason, all the information about the quantum system can be extracted from the Wigner phase space distribution. But it is generally not easy to visualise and even to calculate the Weyl transform of a given operator $\hat{A}(\hat{q},\hat{p})$. For the restricted class of operators

$$\hat{A}_r(\hat{q},\hat{p})=f(\hat{q})+g(\hat{p}) \tag{2.4}$$

the Weyl transform is simply

$$a_r(q,p)=f(q)+g(p) \tag{2.5}$$

and the Wigner phase space distribution immediately gives a picture of the phase space dynamics (by equation (2.3)). The probability distributions of configuration and momentum can be obtained from the Wigner phase space distribution by

$$\langle\psi(t)|q\rangle\langle q|\psi(t)\rangle=\int_{-\infty}^{\infty}\mathrm{d}p\,W(q,p,t) \tag{2.6}$$

$$\langle\psi(t)|p\rangle\langle p|\psi(t)\rangle=\int_{-\infty}^{\infty}\mathrm{d}q\,W(q,p,t). \tag{2.7}$$

This indicates that the Wigner phase space distribution cannot be peaked arbitarily sharp as a function of q and p. Due to Heisenberg's uncertainty relations, a sharply peaked q distribution will give a broad p distribution and vice versa. In § 3 we will start with a coherent state in the bosons, which is a minimum uncertainty state with $\Delta x=\Delta p=1/\sqrt{2}$. A broadening of an initially maximum peaked Wigner phase space distribution reflects the increasing lack of information about the phase space behaviour of the quantum system. If the Hamiltonian is of the form

$$H=p^2/2m+V(q) \tag{2.8}$$

the dynamics of the Wigner phase space distribution is managed by the equation

$$\frac{\partial W}{\partial t}=\{H,W\}+\sum_{k=1}^{\infty}\frac{1}{(2k+1)!}\left(\frac{i}{2}\right)2k\frac{\partial^{2k+1}V}{\partial q^{2k+1}}\frac{\partial^{2k+1}W}{\partial p^{2k+1}}. \tag{2.9}$$

The first term of equation (2.9) is simply the Poisson bracket and moves the initial distribution along the classical trajectories. The second term of equation (2.9) contains partial derivates of odd order higher than three and is therefore a 'diffusion' term. Since the partial derivatives in the diffusion term are higher than two, the Wigner phase space distribution at time t will have negative values even if the initial distribution is positive. Therefore, it cannot be interpreted as a probability distribution, but nevertheless it contains a lot of information about the phase space behaviour of the quantum system.

3. Dynamics of the two-level system in terms of the Wigner matrix

To treat the bosonic degree of freedom of the Hamiltonian (1.1) in phase space, we introduce the Wigner matrix

$$\hat{W}(q,p,t) = W_{11}(q,p,t)|1\rangle\langle 1| + W_{12}(q,p,t)|1\rangle\langle 2| + W_{21}(q,p,t)|2\rangle\langle 1| + W_{22}(q,p,t)|2\rangle\langle 2| \tag{3.1}$$

in σ_x representation. The matrix $\hat{W}\ (q,p,t)$ is explained by the equation

$$|\psi(t)\rangle = |\psi(t)\rangle_1|1\rangle + |\psi(t)\rangle_2|2\rangle \tag{3.2}$$

$$W_{ij}(q,p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\nu\, \mathrm{e}^{\mathrm{i}p\nu}\langle q - \tfrac{1}{2}\nu|\psi(t)\rangle_{ij}\langle\psi(t)|q + \tfrac{1}{2}\nu\rangle \tag{3.3}$$

where $|\psi(t)\rangle$ denotes the state describing the system at time t and $|1\rangle = |\uparrow\rangle_x$, $|2\rangle = |\downarrow\rangle_x$. The expectation value of an arbitrary operator $\hat{A}$ can be calculated using the Weyl transform matrix $a_{ij}\ (q,p)$ defined by

$$a_{ij}(q,p) = \int_{-\infty}^{\infty} \mathrm{d}\nu\, \mathrm{e}^{\mathrm{i}p\nu}\langle i|\langle q - \tfrac{1}{2}\nu|\hat{A}|q + \tfrac{1}{2}\nu\rangle|j\rangle. \tag{3.4}$$

We find

$$\langle\psi(t)|\hat{A}|\psi(t)\rangle = \sum_{i,j=1,2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{d}p\, a_{ij}(q,p)\, W_{ij}(q,p,t). \tag{3.5}$$

Note that for a pure boson operator $\hat{B} = f\ (b, b^+)$ the non-diagonal elements of the Weyl transform matrix $a_{ij}\ (q,p)$ are zero and $a_{11}\ (q,p) = a_{22}\ (q,p)$. For this class of operators we have

$$\langle\psi(t)|\hat{B}|\psi(t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{d}p\, a_{11}(q,p)\{W_{11}(q,p,t) + W_{22}(q,p,t)\}. \tag{3.6}$$

This means that the trace of the Wigner matrix plays the same role for the bosonic degree of freedom of the system under consideration as the Wigner phase space distribution in one-dimensional systems. Therefore the trace of the Wigner matrix gives a complete description of the bosonic behaviour.

Putting $\hat{A} = |\uparrow\rangle_{xx}\langle\uparrow|$, we have

$$a_{11} = 1 \qquad a_{ij} = 0 \qquad \text{for } (i,j) \neq (1,1).$$

This gives

$$\langle\psi(t)|\uparrow\rangle_{xx}\langle\uparrow|\psi(t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{d}p\, W_{11}(q,p,t). \tag{3.7}$$

In the same way we find

$$\langle\psi(t)|\downarrow\rangle_{xx}\langle\downarrow|\psi(t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{d}p\, W_{22}(q,p,t). \tag{3.8}$$

The LHS of equation (3.7) gives the probability of finding the electron at site $|\uparrow\rangle_x$ and the LHS of equation (3.8) gives the probability of finding the electron at site $|\downarrow\rangle_x$. We see that the diagonal elements of the Wigner matrix give only *restricted* information about the fermion. To have complete information about the fermion, the non-diagonal elements of the Wigner matrix are necessary. Since we have discussed the fermion behaviour in detail in I, we will only calculate the diagonal elements and the trace of the Wigner matrix. This is done by using the same complete set of orthonormal functions in Bargmann space as in I:

$$|\phi\rangle_\nu^{(\pm 1)} = \frac{e^{-\kappa^2}}{\sqrt{2}\sqrt{\nu!}}[(\zeta+\sqrt{2}\kappa)^\nu \exp(-\sqrt{2}\kappa\zeta)|\uparrow\rangle_x \pm(-1)^\nu(\zeta-\sqrt{2}\kappa)^\nu \exp\sqrt{2}\kappa\zeta)|\downarrow\rangle_x] \quad (3.9)$$

or in a shorthand notation

$$|\phi\rangle_\nu^{(\pm 1)} = [|f_\nu^\kappa\rangle|\uparrow\rangle_x \pm (-1)^\nu|f_\nu^{-\kappa}\rangle|\downarrow\rangle_x]/\sqrt{2}. \quad (3.10)$$

Here $|f_\nu^\kappa\rangle$ is the νth eigenstate of a displaced harmonic oscillator. The displacement in configuration space is 2κ. Using the expansion

$$|\psi\rangle_n^{(\pm 1)} = \sum_{l=0}^{\infty} \alpha_l^{(n\pm 1)}|\phi\rangle_l^{(\pm 1)} \quad (3.11)$$

of the nth eigenstate of the Hamiltonian (1.1) (as in I) and the initial state

$$|\psi(0)\rangle = \left[\exp(-\alpha^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(\frac{\exp(-\kappa^2)}{\sqrt{n!}}(\zeta+\sqrt{2}\kappa)^n \exp(-\sqrt{2}\kappa\zeta)\right)\right]|\uparrow\rangle_x \quad (3.12)$$

(as in I) one can calculate the time development of the diagonal elements of the Wigner matrix straightforwardly. The result is

$$W_{11}(q,p,t) = \tfrac{1}{4}e^{-\alpha^2}\sum_{n,n'=0}^{\infty}\sum_{\nu,\nu'=0}^{\infty}\sum_{\mu,\mu'=0}^{\infty}\sum_{\gamma,\gamma'=\pm 1}\frac{\alpha^{\mu+\mu'}}{\sqrt{\mu!}\sqrt{\mu'!}}\alpha_\mu^{n,\gamma} \times \alpha_{\mu'}^{n',\gamma'}\exp[-i(E_{n'}^{\gamma'}-E_n^\gamma)t]\alpha_\nu^{n,\gamma}\alpha_{\nu'}^{n',\gamma'}\langle f_\nu^\kappa|\hat{w}(q,p)|f_{\nu'}^\kappa\rangle \quad (3.13)$$

and

$$W_{22}(q,p,t) = \tfrac{1}{4}e^{-\alpha^2}\sum_{n,n'=0}^{\infty}\sum_{\nu,\nu'=0}^{\infty}\sum_{\mu,\mu'=0}^{\infty}\sum_{\gamma,\gamma'=\pm 1}\frac{\alpha^{\mu+\mu'}}{\sqrt{\mu!}\sqrt{\mu'!}}\alpha_\mu^{n,\gamma} \times \alpha_{\mu'}^{n',\gamma'}\exp[-i(E_{n'}^{\gamma'}-E_n^\gamma)t]\alpha_\nu^{n,\gamma}\alpha_{\nu'}^{n',\gamma'}(\gamma\gamma')(-1)^{\nu+\nu'}\langle f_\nu^{-\kappa}|\hat{w}(q,p)|f_{\nu'}^{-\kappa}\rangle \quad (3.14)$$

where the E_n^γ are, of course, the energy eigenvalues of positive and negative parity of the Hamiltonian (1.1) and

$$\hat{w}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dv\, e^{ipv}|q+\tfrac{1}{2}v\rangle\langle q-\tfrac{1}{2}v|. \quad (3.15)$$

The crucial point in the calculation of $W_{11}(q,p,t)$ and $W_{22}(q,p,t)$ is the fact that we are able to find an analytical expression for the Wigner phase space matrix elements $\langle f_\nu^{\pm\kappa}|\hat{w}(q,p)|f_{\nu'}^{\pm\kappa}\rangle$ of the displaced harmonic oscillator. The results of the calculation (see the appendix) can be summarised as follows:

$$\langle f_\nu^{\pm\kappa}|\hat{w}(q,p)|f_{\nu'}^{\pm\kappa}\rangle = \langle f_\nu^0|\hat{w}(q\pm 2\kappa,p)|f_{\nu'}^0\rangle \quad (3.16)$$

and

$$\langle f_\nu^0|\hat{w}(q,p)|f_{\nu'}^0\rangle = \frac{1}{\pi}\exp[i\phi(\nu-\nu')]\left(\frac{\nu!}{\nu'!}\right)^{1/2}(-1)^\nu e^{-r^2}(\sqrt{2}r)^{\nu'-\nu}L_\nu^{\nu'-\nu}(2r^2) \quad (3.17)$$

where r, ϕ are the polar coordinates in phase space:

$$\begin{aligned} q &= r\cos\phi \\ p &= r\sin\phi. \end{aligned} \tag{3.18}$$

The Schrödinger equation in the Wigner matrix formulation is the following system of coupled partial differential equations of first order:

$$\begin{aligned}
\frac{\partial W_{11}}{\partial t} &= (q+2\kappa)\frac{\partial W_{11}}{\partial p} - p\frac{\partial W_{11}}{\partial q} - \frac{\mathrm{i}}{h}\bar{\delta}(W_{21}-W_{12}) \\
\frac{\partial W_{22}}{\partial t} &= (q-2\kappa)\frac{\partial W_{22}}{\partial p} - p\frac{\partial W_{22}}{\partial q} - \frac{\mathrm{i}}{h}\bar{\delta}(W_{12}-W_{21}) \\
\frac{\partial W_{12}}{\partial t} &= q\frac{\partial W_{12}}{\partial p} - p\frac{\partial W_{12}}{\partial q} - \frac{\mathrm{i}}{h}4\kappa q W_{12} - \frac{\mathrm{h}}{h}\bar{\delta}(W_{22}-W_{11}) \\
\frac{\partial W_{21}}{\partial t} &= q\frac{\partial W_{21}}{\partial p} - p\frac{\partial W_{21}}{\partial q} + \frac{\mathrm{i}}{h}4\kappa q W_{21} - \frac{\mathrm{i}}{h}\bar{\delta}(W_{11}-W_{22}).
\end{aligned} \tag{3.19}$$

Equations (3.19) are to be solved with the initial distributions

$$\begin{aligned} W_{11}(q,p,0) &= \pi^{-1}\exp[-(q+2\kappa-\sqrt{2}\alpha)^2 - p^2] \\ W_{ij}(q,p,0) &= 0 \qquad (i,j)\neq(1,1). \end{aligned} \tag{3.20}$$

These initial distributions describe the coherent state (3.12) in the Wigner matrix formulation. The solution of this problem is given by equations (3.13) and (3.14). We have written down equations (3.19) to point out some peculiarities of the two-level system. Normally, the difference of quantum and classical phase space distribution behaviour is due to the 'diffusion' terms in the Schrödinger equation (§ 2). Equations (3.19) do not contain diffusion terms. Since the potential is harmonic, no partial derivates higher than one appear. The complexity of the Wigner matrix motion is due to the coupling, which is switched on by the parameter $\bar{\delta}$. For zero coupling ($\bar{\delta}=0$), the solution with the initial distributions (3.20) is

$$\begin{aligned} W_{11}(q,p,t) &= \pi^{-1}\exp[-(q+2\kappa-\sqrt{2}\alpha\cos t)^2 - (p+\sqrt{2}\alpha\sin t)^2] \\ W_{ij}(q,p,t) &= 0 \qquad (i,j)\neq(1,1). \end{aligned} \tag{3.21}$$

The Gaussian distribution moves simply along the classical trajectories. There is no difference in quantum and classical phase space distribution behaviour.

4. Numerical results

In this paper we use the same parameters as in I. As mentioned before and shown in I, these parameters are well suited to discuss small polaron motion. Therefore we will always have the two-site model of small polaron motion in mind (see I) when we comment on the numerical results.

Figure 1 shows the initial Gaussian distribution (3.20). Since only W_{11} $(q,p,0)$ is different from zero, the trace of the Wigner matrix and W_{11} $(q,p,0)$ are identical. Note that W_{22} $(q,p,0)=0$ means that the electron is at site $|\uparrow\rangle_x$ for $t=0$. Remember that, for $\bar{\delta}=0$, the Gaussian distribution moves along the classical circle $(q+2\kappa)^2+p^2=2\alpha^2$

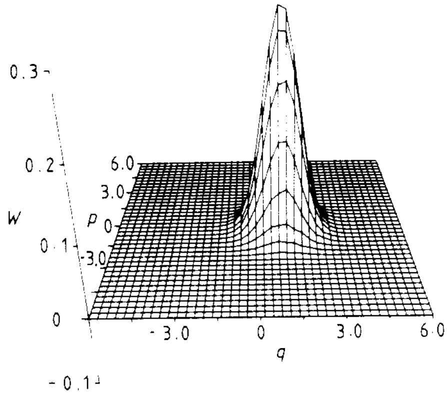

Figure 1. Gaussian phase space distribution W_{11} $(q, p, 0)$ corresponding to the initial coherent state (3.12).

with period $T = 2\pi$ and without changing its shape. We will first discuss the case of small perturbation ($\bar{\delta} = 0.1$). Figure 2(a) shows W_{11} $(q, p, 4T)$. We see a Gaussian-type distribution at the expected place in phase space with a considerably reduced peak height. Since

$$0 = \frac{\partial}{\partial t}\langle\psi(t)|\psi(t)\rangle = \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}q\,\mathrm{d}p\,(W_{11}(q, p, t) + W_{22}(q, p, t))\right) \tag{4.1}$$

or for normalised states

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}p\,\mathrm{d}p\,(W_{11}(q, p, t) + W_{22}(q, p, t)) = 1 \tag{4.2}$$

the distribution W_{22} $(q, p, 4T)$ must be different from zero. The electron is no longer at site $|\uparrow\rangle_x$, but it is 'between' the two sites. The structure of W_{22} $(q, p, 4T)$, which is shown in figure 2(b), is a very typical one for times up to $10T$. The sharp peak in the structure of W_{22} (q, p, t) moves along the circle $(q - 2\kappa)^2 + p^2 = 2\alpha^2$ with period $T = 2\pi$. Only the height of the whole structure changes considerably in time. The superposition of W_{11} $(q, p, 4T)$ and W_{22} $(q, p, 4T)$ gives the trace of the Wigner matrix at $t = 4T$ (see

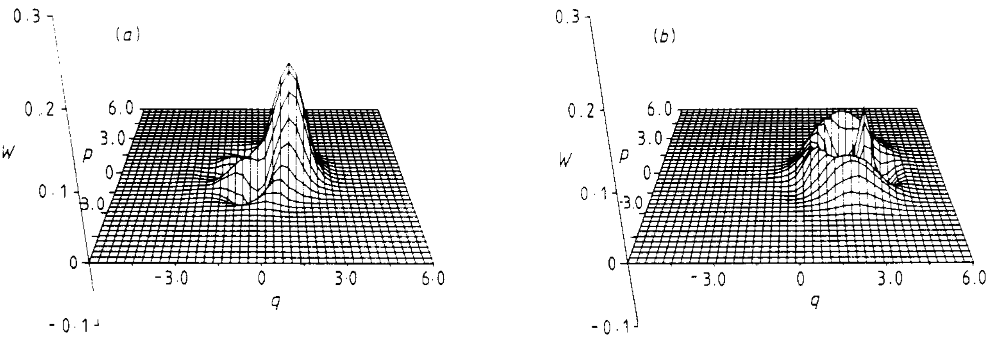

Figure 2. (a) Wigner matrix element W_{11} $(q, p, 4T)$. (b) Wigner matrix element W_{22} $(q, p, 4T)$. The parameters used are $\kappa = 0.5$, $\bar{\delta} = 0.1$, $\alpha = 1.5$.

figure 3(a)). In order to save space, we present for the times $t=6T$ (figure 3(b)), $t=8T$ (figure 3(c)) and $t=10T$ (figure 3(d)) only the trace of the Wigner matrix, which gives complete information about the bosonic degree of freedom as mentioned before. The striking features of these pictures are the strongly peaked structures enclosed by a broad shoulder. Since the trace of the Wigner operator has negative values, the relevance of these strongly peaked structures is not clear. This is because they can cancel each other away in the calculation of the integrals (2.6), (2.7) or (3.6). To demonstrate this, we have integrated $\mathrm{Tr}(\hat{W}(q, p, 10T))$ with respect to p to obtain the probability distribution in configuration space (see figure 9(a)) and with respect to q to obtain the probability distribution in momentum space (see figure 4(b)) of the bosonic degree of freedom. We see that the peaked structures are reflected in the configuration distribution but not in the momentum distribution. Due to the rotation of $W_{11}(q, p)$ and $W_{22}(q, p)$ the role of configuration and momentum in respect to the peaked structures is changed in time.

Let us now turn to the case of resonance ($\bar{\delta}=\frac{1}{2}$). The most striking difference to the case of small perturbation is the fact that the fermion needs only one period T to destroy almost completely the initial coherent state (see figure 5(a)). Figures 5(b)–(d) show the trace of the Wigner matrix for the successive times $3/2T$, $2T$ and $5/2T$. Note that, for resonance $\mathrm{Tr}(\hat{W}(q, p, t))$ changes much more rapidly in time than for small perturbation. It is interesting that $\mathrm{Tr}(\hat{W}(q, p, t))$ for $\bar{\delta}=0.5$ is similar to $\mathrm{Tr}(\hat{W}(q, p, 5t))$ for $\bar{\delta}=0.1$ (compare, for example, figures 3(d) and 5(c). This suggests that for the five times greater perturbation the dynamics is very similar to the dynamics of small perturbation but on a timescale which differs from the previous one by a factor of five.

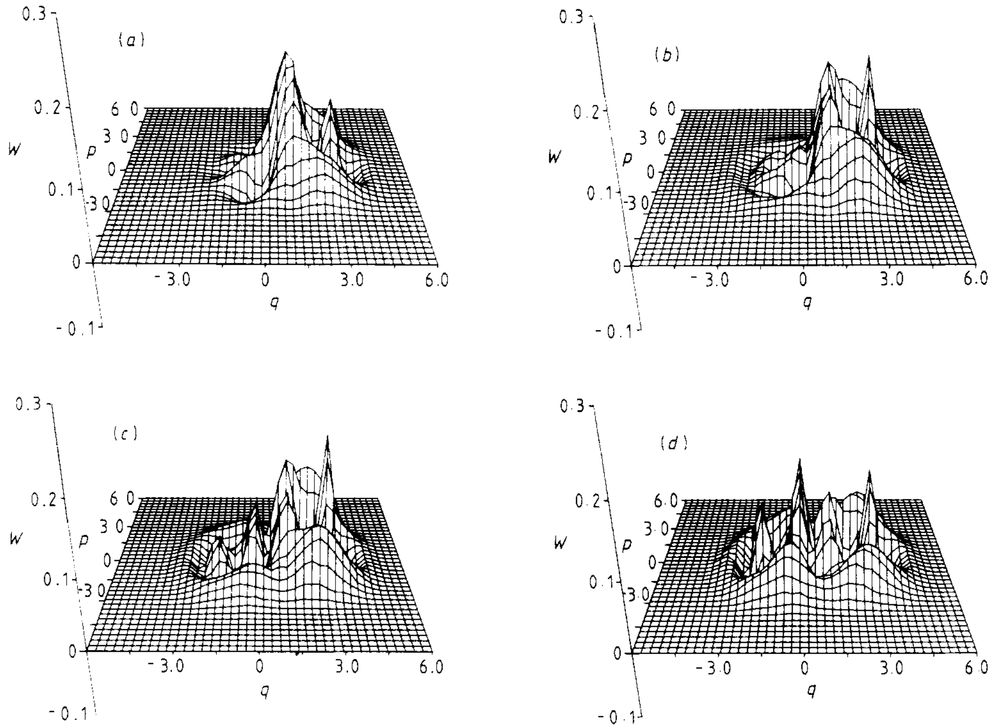

Figure 3. Trace of the Wigner matrix at time t: (a), $4T$; (b), $6T$; (c), $8T$; (d) $10T$. Parameters as in figure 2.

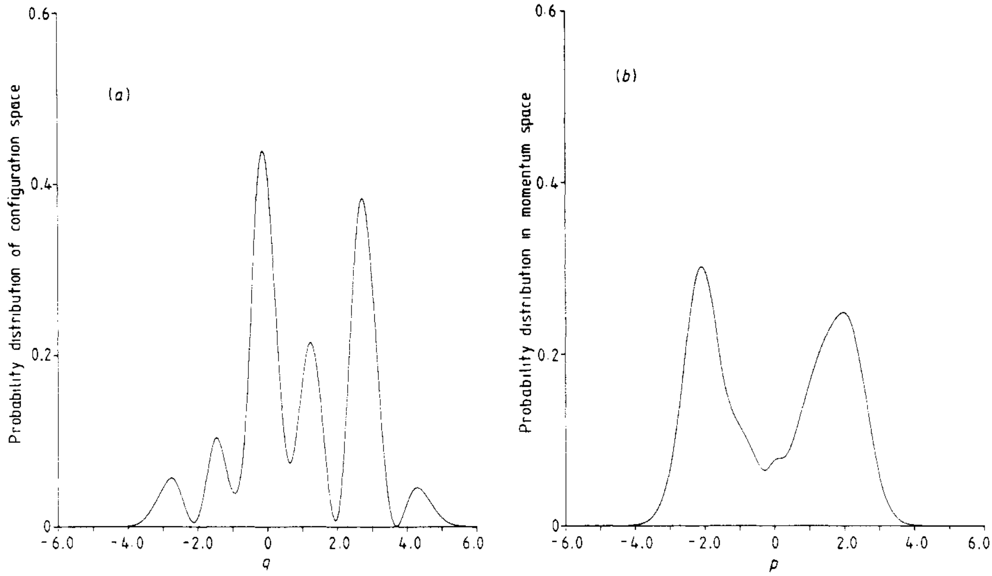

Figure 4. Probability distribution in (*a*), configuration space; (*b*), momentum space at time $t = 10T$. Parameters as in figure 2.

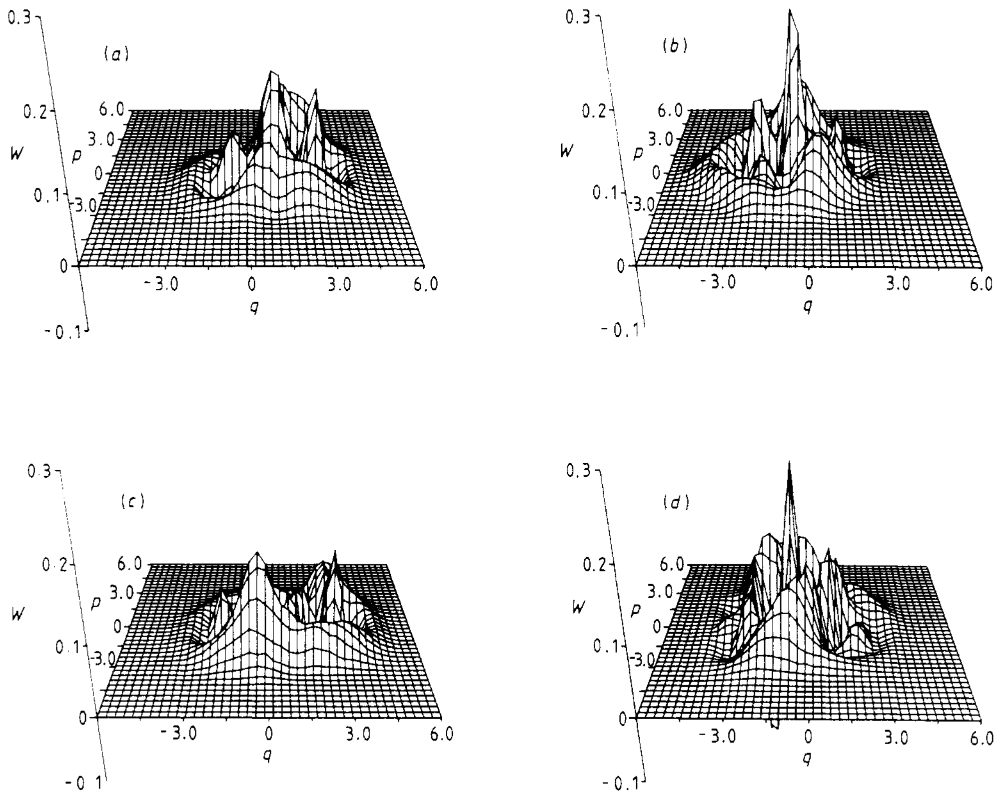

Figure 5. Trace of the Wigner matrix at time t: (*a*), T; (*b*), $3/2T$; (*c*), $2T$; (*d*), $5/2T$. The parameters are $\kappa = 0.5$, $\bar{\delta} = 0.5$, $\alpha = 1.5$.

We have observed this behaviour up to times for which $\bar{\delta}t < 10$. For greater times, this correspondence breaks down and it is hard to see any regularity in the motion.

Until now, we have presented the dynamics of the two-level system for small times. Our next aim is to obtain some information about the long time behaviour of this system. For this purpose, we have calculated the long time average of the diagonal elements and the trace of the Wigner matrix. Since the energy spectrum is non-degenerate for the parameters under consideration we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp[-\mathrm{i}(E_{n'}^{\gamma'} - E_n^{\gamma})t]\,\mathrm{d}t = \delta_{n'n}\delta_{\gamma\gamma'}. \tag{4.3}$$

The long time average is now obtained when the oscillating terms in (3.13) and (3.14) are replaced by the Kronecker deltas. It is sufficient to calculate the long time average

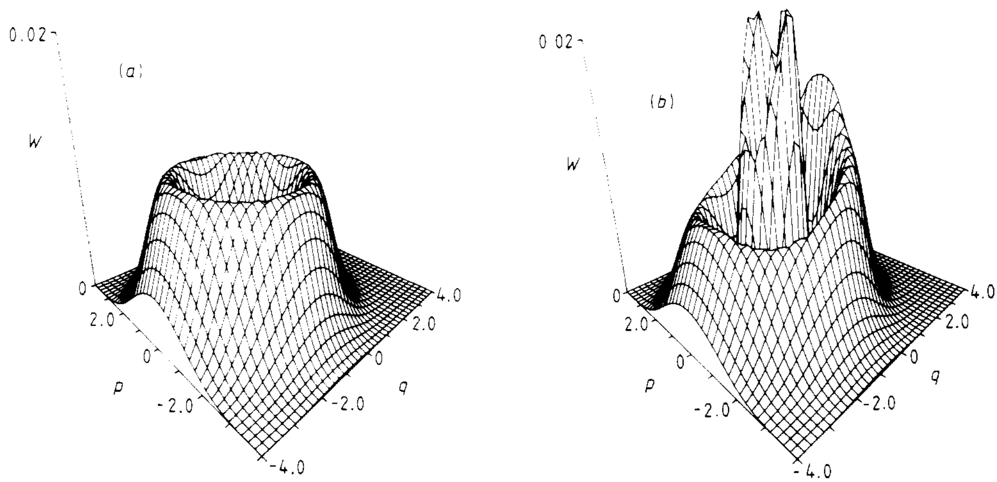

Figure 6. (*a*) Long time average of the Wigner matrix element W_{11} (q, p, t), parameters as in figure 2; (*b*) for $\bar{\delta} = 0.5$.

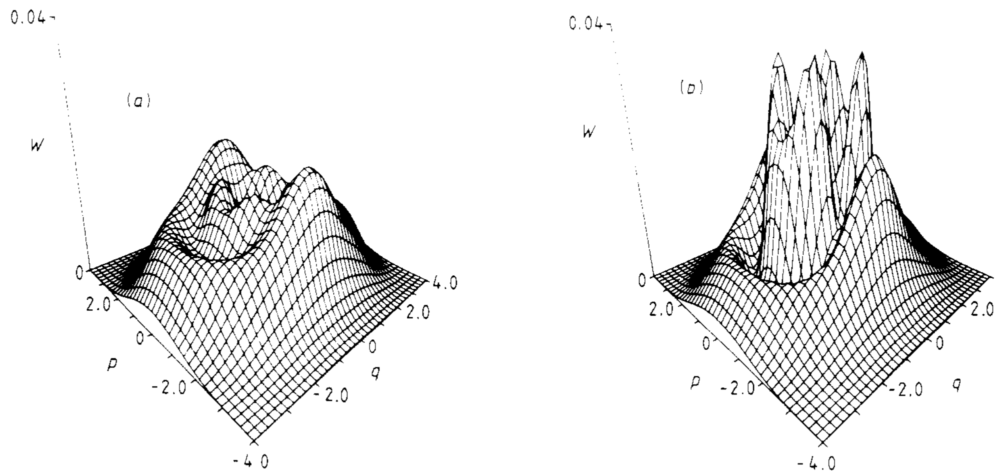

Figure 7. (*a*) Long time average of the trace of the Wigner matrix, parameters as in figure 2; (*b*) for $\bar{d} = 0.5$.

$\bar{W}_{11}\,(q,p)$ because

$$\bar{W}_{11}(q,p)=\bar{W}_{22}(-q,-p) \tag{4.4}$$

as can be seen from (3.13), (3.14) and (4.3).

Figure 6(a) shows the behaviour of the bosonic degree of freedom averaged over the times when the electron is at or near the site $|\uparrow\rangle_x$. In the long time average, the peaked structures cancel each other partly away for small perturbations. The harmonic 'torus' dominates in this case. This situation is changed for the case of resonance (see figure 6(b)). The harmonic 'torus' is damaged as the peaked structures in the middle increase in height. The structure of the long time average of the trace of the Wigner matrix in both cases (see figure 7) can be understood as the supperposition of the two displaced, relatively simple structured diagonal elements. It is interesting that the results are very similar in both cases, when only the structure is taken into account. The main difference is the relative height of inner and outer structures. This is due to the behaviour of the expansion coefficients $\alpha_l^{(n\pm1)}$ which are sharply peaked in both cases (see tables 1 and 2 of I).

5. Discussion

We have presented in this paper the quantum dynamics of the bosonic degree of freedom of the two-level system in detail. We have used a Wigner matrix formulation, introduced in § 3, because it gives information about some qualitative features of the bosonic phase space motion. We now consider the implications of our results for the 'quantum chaos' of the two-level system. We have shown in I that the semiclassical and quantum trajectories are not correlated for the parameters used. For this reason all the chaotic properties of the semiclassical trajectories are not relevant in the quantum case. Only for increasing excitation of the bosons will chaotic features of the semiclassical motion influence the quantum motion, due to the correspondence principle. Next we can ask about chaotic properties of the quantum system without referring to the semiclassical limit. We have seen that the time evolution of the trace of the Wigner matrix is complex and shows hardly any regularity. But it seems that this is not a measure of 'quantum chaos'. We have shown in I that the criterion of quantum integrability of Hose and Taylor (1983) is satisfied up to resonance for all couplings. This means that the whole energy spectrum is regular up to resonance. Additionally, Kus (1985) has observed regularities in the nearest-neighbour level spacings. This led him to the conclusion that there is no quantum chaos in the two-level system, although Farrelly (1984) and Berry (1984) showed that this distribution can be irregular for integrable cases.

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Appendix. Calculation of the Wigner phase space matrix elements $\langle f_\nu^\kappa|w(q,p)|f_{\nu'}^\kappa\rangle$ of the displaced harmonic oscillator

Since $|f_{\nu'}^\kappa\rangle$ is the ν'th eigenstate of the displaced harmonic oscillator (the displacement in configuration space is 2κ), we have

$$\langle q|f_{\nu'}^{\kappa}\rangle = \langle q+2\kappa|f_{\nu'}^{0}\rangle \tag{A1}$$

and

$$\langle f_{\nu}^{\kappa}|q\rangle = \langle f_{\nu}^{0}|q+2\kappa\rangle. \tag{A2}$$

Equations (A1) and (A2) lead to

$$\begin{aligned}\langle f_{\nu}^{\kappa}|\hat{w}(q,p)|f_{\nu'}^{\kappa}\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}pv}\langle f_{\nu}^{\kappa}|q+\tfrac{1}{2}v\rangle\langle q-\tfrac{1}{2}v|f_{\nu'}^{\kappa}\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}pv}\langle f_{\nu}^{0}|q+2\kappa+\tfrac{1}{2}v\rangle\langle q+2\kappa-\tfrac{1}{2}v|f_{\nu'}^{0}\rangle \\ &= \langle f_{\nu}^{0}|\hat{w}|q+2\kappa, p)|f_{\nu'}^{0}\rangle. \end{aligned} \tag{A3}$$

Therefore it is sufficient to calculate the Wigner phase space matrix elements for the harmonic oscillator. To do this, we calculate first the Wigner phase space distribution $T(q, p, \xi, \eta)$ for the generating function

$$\Omega(q,\zeta) = \exp(-q^2/2)\exp(-\zeta^2+2\zeta q) = \sum_{\nu=0}^{\infty} \pi^{1/4}(2^{\nu}/\nu!)^{1/2}\zeta^{\nu}\langle q|f_{\nu}^{0}\rangle \tag{A4}$$

of the harmonic oscillator:

$$T(q,p,\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}2y}\Omega^*(q-y,\xi)\Omega(q+y,\eta). \tag{A5}$$

In contrast to the integrals arising from the Wigner phase space matrix elements of the harmonic oscillator, the integral (A5) of the generating function $\Omega(q,\xi)$ can be calculated straightforwardly. The result is

$$T(q,p,\xi,\eta) = \exp(-q^2-p^2)\exp[2\xi^*(q+\mathrm{i}p-\eta)]\exp[2\eta(q-\mathrm{i}p)]/\sqrt{\pi}. \tag{A6}$$

On the other hand, the RHS of (A4) gives

$$T(q,p,\xi,\eta) = \sum_{\nu,\nu'=0}^{\infty} \sqrt{\pi}\left(\frac{2^{\nu}}{\nu!}\frac{2^{\nu'}}{\nu'!}\right)^{1/2} \xi^{*\nu}\eta^{\nu'}\langle f_{\nu}^{0}|\hat{w}(q,p)|f_{\nu'}^{0}\rangle. \tag{A7}$$

Next we expand the two last exponential functions in (A6) and equate the coefficients of $\xi^{*\nu}\eta^{\nu'}$ in (A6) and (A7). We find after some algebra that

$$\langle f_{\nu}^{0}|\hat{w}(q,p)|f_{\nu'}^{0}\rangle = \frac{\exp[\mathrm{i}\phi(\nu-\nu')]}{\pi}\left(\frac{\nu!}{\nu'!}\right)^{1/2}(-1)^{\nu}\,\mathrm{e}^{-r^2}(\sqrt{2}r)^{\nu'-\nu}L_{\nu}^{\nu'-\nu}(2r^2) \tag{A8}$$

where r, ϕ are the polar coordinates in phase space.

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